Transition Maths and Algebra with Geometry

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Lecture Notes Electrical and Computer Engineering









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3 Nonhomogeneous and associated homogeneous systems







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System of linear equations

Definition

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots a_{1n} \cdot x_n = b_1,$$

 $a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots a_{2n} \cdot x_n = b_2,$
 \dots

$$a_{m1}\cdot x_1+a_{m2}\cdot x_2+\ldots a_{mn}\cdot x_n=b_m.$$

is a system of *m* linear equations with *n* uknowns: $x_1, \ldots x_n$.

Definition

A solution of the above system is a set of values for the unknowns $x_1 = k_1, \ldots, x_n = k_n$ which satisfy the above *m* equalities.

Example of SoLE

$$3x + y + z = 1,$$
$$x + y = 0.$$

This is a system of 2 equations with 3 unknowns. It is easy to check that x = 0, y = 0, z = 1 or x = -1, y = 1, z = 3 is a solution (there are other!).

$$\begin{aligned} x + y &= 1, \\ x + y &= 0. \end{aligned}$$

This is a system of 2 equations with 2 unknowns. It has no solutions since if there were any then it would mean that 0 = 1.



Matrix representation

Observation

A system of *m* linear equations with x_1, \ldots, x_n as unknowns:

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n = b_1,$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n = b_2,$$

$$\dots$$

$$a_{m1} \cdot x_1 + a_{k2} \cdot x_2 + \ldots + a_{mn} \cdot x_n = b_m.$$

can be represented by a matrix equation:

 $A \cdot X = B$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & \dots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

Equivalence

Definition

Two systems $A_1X = B_1$ and $A_2X = B_2$ of *m* linear equations with *n* unknowns are said to be *equivalent* if they have the same solution set.

Example:

$$\begin{aligned} x - y &= 1, \\ x + y &= 0. \end{aligned}$$

is equivalent to

$$2x - 2y = 2,$$
$$x + y = 0.$$



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Augmented matrix

Definition

The matrix

$$(A|B) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

is called an augmented matrix of the system $A \cdot X = B$.

Example: the matrix

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$$\left(\begin{array}{cc|c} 2 & -2 & 2\\ 1 & 1 & 0 \end{array}\right)$$

is the augmented matrix of the system

$$2x - 2y = 2$$



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Equivalence of systems: properties

Given two systems $A_1X = B_1$ and $A_2X = B_2$ when are they equivalent? How

can we show that they have the same solution set?

Theorem

Two systems $A_1X = B_1$ and $A_2X = B_2$ of *m* linear equations with *n* unknowns are equivalent iff their augmented matrices $(A_1|B_1)$ and $(A_2|B_2)$ are row equivalent.

Recall that two matrices are row equivalent if one can be obtained from the other by elementary row operations:

- **(Row switching)** *i*-th row and *j*-th row are interchanged $(R_i \leftrightarrow R_j)$,
- **(Row scaling)** each element in *i*-th row is multiplied by a *nonzero* scalar $k \in \mathbb{K} (kR_i \rightarrow R_i)$,
- **(Row addition)** *i*-th row is replaced by a sum of *i*-th row and a multiple of *j*-th row $(R_i + k \cdot R_i \rightarrow R_i)$.

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SoLEs in echelon form

Definition

A system AX = B of linear equations is said to be in *echelon form* if the augmented matrix (A|B) is in row echelon form.

Recall that a matrix is in row echelon form if

- all zero rows are at the bottom,
- Ithe first nonzero number in the *i*-th row is to the right from the first nonzero coefficient in the row above it.









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Solving SoLEs

Recall that each matrix is row equivalent to a matrix in row echelon form. Therefore, for any system AX = B there exists an equivalent system A'X = B' in echelon form. In other words, there is a system A'X = B' in echelon form with the same solution set.

Solving SoLEs

- Given a system AX = B write its augmented matrix (A|B),
- Find a matrix (A'|B') in row echelon form which is row equivalent to (A|B),
- Write the new system A'X = B' and deduce its solutions.



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Solving SoLEs: Examples

Consider the following SoLE:

$$2x + 4y - z = 11,$$

-4x - 3y + 3z = 20,
2x + 4y + 2z = 2.

Its augmented matrix is given by:



Solving SoLEs: Examples

Using elementary row operations we obtain the following matrix in row echelon form equivalent to the matrix from the previous slide:









Solving SoLEs: Examples

$$\left(\begin{array}{ccccc} 2 & 4 & -1 & | & 11 \\ 0 & 5 & 1 & | & 2 \\ 0 & 0 & 3 & | & -9 \end{array}\right)$$

This is the augmented matrix of the system:

$$2x + 4y - z = 11,$$

$$5y + z = 2,$$

$$3z = -9.$$

We see that the 3rd equation implies that z = -3. This and the 2nd equations gives us 5y - 3 = 2 and hence y = 1. Similarly we show that x = 2. The system has a unique solution x = 2, y = 1, z = -3.



Solving SoLEs: Examples

Consider the following SoLE:

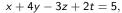
$$x + 4y - 3z + 2t = 5,$$

 $2x + 8y - 5z = 12.$

It augmented matrix is:

By row reduction we get:







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Solving SoLEs: Examples

$$x + 4y - 3z + 2t = 5,$$
$$z - 4t = 2.$$

This system in echelon form doesn't have a triangular form as in the previous example. We see that there are 2 equations and 4 unknowns. The system has more than one solution. We take non-leading variables, namely y and t, and assign arbitrary values to these. Say y = a and t = b. Then we use a simple substitution to get: z - 4b = 2 hence z = 2 + 4b and x = 11 - 4a + 10b. The solution depends on two parameters, a and b and is given by:

$$x = 11 - 4a + 10b$$
, $y = a$, $z = 2 + 4b$, $t = b$



Solving SoLEs: Examples

Consider the following SoLE:

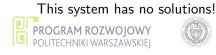
$$2x + 4y = 0,$$
$$2x + 4y = 1.$$

Its augmented matrix is given by:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 1 \end{array}\right)$$

Using elementary row operations we obtain the following matrix in row echelon form equivalent to the matrix above:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 1 \end{array}\right)$$





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Kronecker-Capelli Theorem

Theorem

A system AX = B of linear equations has a solution iff r(A) = r(A|B).

Example:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 1 \end{array}\right)$$







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3 Nonhomogeneous and associated homogeneous systems









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Homogeneous SoLEs

Definition

A system of linear equations is called $\ensuremath{\textit{homogeneous}}$ if it is of the form

$$A \cdot X = 0$$

The homogeneous system of linear equations ALWAYS has a solution, namely $x_1 = 0, x_2 = 0, ..., x_n = 0$. Any other solution (if there is one) is called a *non-trivial* solution.



Homogeneous SoLEs:Examples

Consider the following system:

$$x + 2y - 3z + w = 0$$
$$x - 3y + z - 2w = 0$$
$$2x + y - 3z + 5w = 0$$

We see that there are 3 equations and 4 unknowns. Therefore, there are non-trivial solutions!



Homogeneous SoLEs:Examples

Consider the following HSoLE:

$$x + y - z = 0$$
$$2x - 3y + z = 0$$
$$x - 4y + 2z = 0.$$

It can be reduced to

$$\begin{aligned} x + y - z &= 0\\ -5y + 3z &= 0. \end{aligned}$$

This system also has a non-trivial solution since if we take the non-leading variable z and assign to it any value, say z = a, then $y = \frac{3}{5}a$ and $x = \frac{2}{5}a$.



Homogeneous SoLEs:Examples

Consider the following HSoLE:

$$x + y - z = 0$$
$$2x + 4y - z = 0$$
$$3x + 2y + 2z = 0.$$

It can be reduced to

$$x + y - z = 0,$$

$$2y + z = 0,$$

$$11z = 0.$$

This system also has only the zero solution.



Solution space

Theorem

Let AX = 0 be a system of *m* linear homogeneous equations with *n* unknowns. Then the set $W = \{ \mathbf{v} \in \mathbb{K}^n \mid A\mathbf{v} = 0 \}$ of solutions is a subspace of the vector space \mathbb{K}^n . Moreover,

$$\dim(W) = n - r(A).$$

Proof (of the 1st part of the statement): Take $\mathbf{v}_1, \mathbf{v}_2 \in W$. This means that $A\mathbf{v}_1 = 0$ and $A\mathbf{v}_2 = 0$. Hence,

$$A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2 = 0 + 0 = 0.$$

Therefore, $\mathbf{v_1} + \mathbf{v_2} \in W$. Similarly, we prove that for any $k \in \mathbf{K}$, $k \cdot \mathbf{v_1} \in W$.



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Solution space and basis: Example

Consider the following HSoLE:

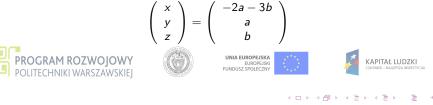
$$x + 2y + 3z = 0,$$

 $-2x - 4y - 6z = 0.$

It reduces to:

$$x + 2y + 3z = 0.$$

The non-leading variables are y and z. Put y = a and z = b. The solutions are of the form:



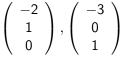
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Solution space and basis: Example

The solution space is given by:

$$W = \{ \left(egin{array}{c} -2a-3b\ a\ b \end{array}
ight) \mid a,b\in \mathbb{R} \}.$$

We see that any vector from W depends on two parameters, namely a and b. If we put a = 1, b = 0 and a = 0, b = 1 we will obtain two vectors



which form a basis of *W*. **PROGRAM ROZWOJOWY** POLITECHNIKI WARSZAWSKIEJ







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Homogeneous and nonhomogeneous systems

Theorem

Let AX = B be an arbitrary system of linear equations. Let U be the solution set and let $\mathbf{v}_0 \in U$ be a particular solution to this system of linear equations. Then

$$U = \mathbf{v}_0 + W = \{\mathbf{v}_0 + \mathbf{w} \mid \mathbf{w} \in W\},\$$

where W is the solution space of the homogeneous system AX = 0.

Proof: any element from $\mathbf{v}_0 + W$ is a solution to AX = B. Indeed,

$$A(\mathbf{v}_0 + \mathbf{w}) = A\mathbf{v}_0 + A\mathbf{w} = B + 0 = B.$$

Moreover, if $A\mathbf{v} = B$ then put $w = \mathbf{v} - \mathbf{v}_0$. We see that $\mathbf{v} = \mathbf{v}_0 + \mathbf{w}$ and



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