Transition Maths and Algebra with Geometry

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Lecture Notes
Electrical and Computer Engineering

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System of linear equations

Definition

$$
\begin{aligned}
a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots a_{1n} \cdot x_n &= b_1, \\
a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots a_{2n} \cdot x_n &= b_2, \\
&\ldots \\
a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \ldots a_{mn} \cdot x_n &= b_m.
\end{aligned}
$$

is a system of m linear equations with n uknowns: $x_1, \ldots x_n$.

Definition

A solution of the above system is a set of values for the unknowns $x_1 = k_1, \ldots, x_n = k_n$ which satisfy the above m equalities.

Example of SoLE

$$\begin{aligned} 3x + y + z &= 1, \\ x + y &= 0. \end{aligned}$$

This is a system of 2 equations with 3 unknowns. It is easy to check that $x = 0, y = 0, z = 1$ or $x = -1, y = 1, z = 3$ is a solution (there are other!).

$$\begin{aligned} x + y &= 1, \\ x + y &= 0. \end{aligned}$$

This is a system of 2 equations with 2 unknowns. It has no solutions since if there were any then it would mean that $0 = 1$.

Matrix representation

Observation

A system of m linear equations with $x_1, \ldots, x_n$ as unknowns:

$$\begin{aligned} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n &= b_1, \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n &= b_2, \\ &\ldots \\ a_{m1} \cdot x_1 + a_{k2} \cdot x_2 + \ldots + a_{mn} \cdot x_n &= b_m. \end{aligned}$$

can be represented by a matrix equation:

$$A \cdot X = B$$

where

$$A = \left(\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & \ldots & \ldots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{array} \right) X = \left(\begin{array}{c} x_1 \\ x_2 \\ \ldots \\ x_n \end{array} \right) B = \left(\begin{array}{c} b_1 \\ b_2 \\ \ldots \\ b_m \end{array} \right)$$

Equivalence

Definition

Two systems $A_1X = B_1$ and $A_2X = B_2$ of m linear equations with n unknowns are said to be *equivalent* if they have the same solution set.

Example:

$$x - y = 1,$$
$$x + y = 0.$$

is equivalent to

$$2x - 2y = 2,$$
$$x + y = 0.$$

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Augmented matrix

Definition

The matrix

$$(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

is called an augmented matrix of the system $A \cdot X = B$.

Example: the matrix

$$\left(\begin{array}{cc|c} 2 & -2 & 2 \\ 1 & 1 & 0 \end{array}\right)$$

is the augmented matrix of the system

$$2x - 2y = 2,$$
$$x + y = 0.$$

Equivalence of systems: properties

Given two systems $A_1X = B_1$ and $A_2X = B_2$ when are they equivalent? How can we show that they have the same solution set?

Theorem

Two systems $A_1X = B_1$ and $A_2X = B_2$ of m linear equations with n unknowns are equivalent iff their augmented matrices $(A_1|B_1)$ and $(A_2|B_2)$ are row equivalent.

Recall that two matrices are row equivalent if one can be obtained from the other by elementary row operations:

1. **(Row switching)** i-th row and j-th row are interchanged ($R_i \leftrightarrow R_j$),
2. **(Row scaling)** each element in i-th row is multiplied by a *nonzero* scalar $k \in \mathbb{K}$ ($kR_i \to R_i$),
3. **(Row addition)** i-th row is replaced by a sum of i-th row and a multiple of j-th row ($R_i + k \cdot R_j \to R_i$).

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SoLEs in echelon form

Definition

A system $AX = B$ of linear equations is said to be in *echelon form* if the augmented matrix $(A|B)$ is in row echelon form.

Recall that a matrix is in *row echelon form* if

1. all zero rows are at the bottom,
2. the first nonzero number in the i-th row is to the right from the first nonzero coefficient in the row above it.

Solving SoLEs

Recall that each matrix is row equivalent to a matrix in row echelon form. Therefore, for any system $AX = B$ there exists an equivalent system $A'X = B'$ in echelon form. In other words, there is a system $A'X = B'$ in echelon form with the same solution set.

Solving SoLEs

- Given a system $AX = B$ write its augmented matrix $(A|B)$,
- Find a matrix $(A'|B')$ in row echelon form which is row equivalent to $(A|B)$,
- Write the new system $A'X = B'$ and deduce its solutions.

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Solving SoLEs: Examples

Consider the following SoLE:

$$\begin{aligned} 2x + 4y - z &= 11, \\ -4x - 3y + 3z &= 20, \\ 2x + 4y + 2z &= 2. \end{aligned}$$

Its augmented matrix is given by:

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 11 \\ -4 & -3 & 3 & 20 \\ 2 & 4 & 2 & 2 \end{array}\right)$$

Solving SoLEs: Examples

Using elementary row operations we obtain the following matrix in row echelon form equivalent to the matrix from the previous slide:

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 11 \\ 0 & 5 & 1 & 2 \\ 0 & 0 & 3 & -9 \end{array}\right)$$

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Solving SoLEs: Examples

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 11 \\ 0 & 5 & 1 & 2 \\ 0 & 0 & 3 & -9 \end{array}\right)$$

This is the augmented matrix of the system:

$$\begin{aligned} 2x + 4y - z &= 11, \\ 5y + z &= 2, \\ 3z &= -9. \end{aligned}$$

We see that the 3rd equation implies that $z = -3$. This and the 2nd equations gives us $5y - 3 = 2$ and hence $y = 1$. Similarily we show that $x = 2$. The system has a unique solution $x = 2, y = 1, z = -3$.

Solving SoLEs: Examples

Consider the following SoLE:

$$\begin{aligned} x + 4y - 3z + 2t &= 5, \\ 2x + 8y - 5z &= 12. \end{aligned}$$

It augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 4 & -3 & 2 & 5 \\ 2 & 8 & -5 & 0 & 12 \end{array}\right)$$

By row reduction we get:

$$\left(\begin{array}{cccc|c} 1 & 4 & -3 & 2 & 5 \\ 0 & 0 & 1 & -4 & 2 \end{array}\right)$$

$$\begin{aligned} x + 4y - 3z + 2t &= 5, \\ z - 4t &= 2. \end{aligned}$$

Solving SoLEs: Examples

$$\begin{aligned} x + 4y - 3z + 2t &= 5, \\ z - 4t &= 2. \end{aligned}$$

This system in echelon form doesn't have a triangular form as in the previous example. We see that there are 2 equations and 4 unknowns. The system has more than one solution. We take non-leading variables, namely y and t, and assign arbitrary values to these. Say $y = a$ and $t = b$. Then we use a simple substitution to get: $z - 4b = 2$ hence $z = 2 + 4b$ and $x = 11 - 4a + 10b$. The solution depends on two parameters, a and b and is given by:

$$x = 11 - 4a + 10b, \;\; y = a, \;\; z = 2 + 4b, \;\; t = b.$$

Solving SoLEs: Examples

Consider the following SoLE:

$$2x + 4y = 0,$$
$$2x + 4y = 1.$$

Its augmented matrix is given by:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 1 \end{array}\right)$$

Using elementary row operations we obtain the following matrix in row echelon form equivalent to the matrix above:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

This system has no solutions!

Kronecker-Capelli Theorem

Theorem

A system $AX = B$ of linear equations has a solution iff $r(A) = r(A|B)$.

Example:

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

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Homogeneous SoLEs

Definition

A system of linear equations is called *homogeneous* if it is of the form

$$A \cdot X = 0$$

The homogeneous system of linear equations ALWAYS has a solution, namely $x_1 = 0, x_2 = 0, \dots, x_n = 0$. Any other solution (if there is one) is called a *non-trivial* solution.

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Homogeneous SoLEs:Examples

Consider the following system:

$$
\begin{aligned}
x + 2y - 3z + w &= 0 \\
x - 3y + z - 2w &= 0 \\
2x + y - 3z + 5w &= 0
\end{aligned}
$$

We see that there are 3 equations and 4 unknowns. Therefore, there are non-trivial solutions!

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Homogeneous SoLEs:Examples

Consider the following HSoLE:

$$\begin{aligned} x + y - z &= 0 \\ 2x - 3y + z &= 0 \\ x - 4y + 2z &= 0. \end{aligned}$$

It can be reduced to

$$\begin{aligned} x + y - z &= 0 \\ -5y + 3z &= 0. \end{aligned}$$

This system also has a non-trivial solution since if we take the non-leading variable z and assign to it any value, say $z = a$, then $y = \frac{3}{5}a$ and $x = \frac{2}{5}a$.

Homogeneous SoLEs:Examples

Consider the following HSoLE:

$$\begin{aligned} x + y - z &= 0 \\ 2x + 4y - z &= 0 \\ 3x + 2y + 2z &= 0. \end{aligned}$$

It can be reduced to

$$\begin{aligned} x + y - z &= 0, \\ 2y + z &= 0, \\ 11z &= 0. \end{aligned}$$

This system also has only the zero solution.

Solution space

Theorem

Let $AX = 0$ be a system of m linear homogeneous equations with n unknowns. Then the set $W = \{\mathbf{v} \in \mathbb{K}^n \mid A\mathbf{v} = 0\}$ of solutions is a subspace of the vector space $\mathbb{K}^n$. Moreover,

$$\dim(W) = n - r(A).$$

Proof (of the 1st part of the statement):
Take $\mathbf{v_1}, \mathbf{v_2} \in W$. This means that $A\mathbf{v_1} = 0$ and $A\mathbf{v_2} = 0$. Hence,

$$A(\mathbf{v_1} + \mathbf{v_2}) = A\mathbf{v_1} + A\mathbf{v_2} = 0 + 0 = 0.$$

Therefore, $\mathbf{v_1} + \mathbf{v_2} \in W$. Similarily, we prove that for any $k \in \mathbf{K}$, $k \cdot \mathbf{v_1} \in W$.

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Solution space and basis: Example

Consider the following HSoLE:

$$\begin{aligned} x + 2y + 3z &= 0, \\ -2x - 4y - 6z &= 0. \end{aligned}$$

It reduces to:

$$x + 2y + 3z = 0.$$

The non-leading variables are y and z. Put $y = a$ and $z = b$. The solutions are of the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2a - 3b \\ a \\ b \end{pmatrix}$$

Solution space and basis: Example

The solution space is given by:

$$W = \{ \begin{pmatrix} -2a-3b \\ a \\ b \end{pmatrix} \mid a, b \in \mathbb{R} \}.$$

We see that any vector from W depends on two parameters, namely a and b. If we put $a=1, b=0$ and $a=0, b=1$ we will obtain two vectors

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

which form a basis of W.

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Homogeneous and nonhomogeneous systems

Theorem

Let $AX = B$ be an arbirary system of linear equations. Let U be the solution set and let $\mathbf{v}_0 \in U$ be a particular solution to this system of linear equations. Then

$$U = \mathbf{v}_0 + W = \{\mathbf{v}_0 + \mathbf{w} \mid \mathbf{w} \in W\},$$

where W is the solution space of the homogeneous system $AX = 0$.

Proof: any element from $\mathbf{v}_0 + W$ is a solution to $AX = B$. Indeed,

$$A(\mathbf{v}_0 + \mathbf{w}) = A\mathbf{v}_0 + A\mathbf{w} = B + 0 = B.$$

Moreover, if $A\mathbf{v} = B$ then put $w = \mathbf{v} - \mathbf{v}_0$. We see that $\mathbf{v} = \mathbf{v}_0 + \mathbf{w}$ and

$$A\mathbf{w} = A(\mathbf{v} - \mathbf{v}_0) = A\mathbf{v} - A\mathbf{v}_0 = B - B = 0.$$